Universal law for axisymmetric MHD turbulence

Loi universelle pour la turbulence MHD axisymétrique

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(Dated: March 8, 2010)

Abstract

Homogeneous MHD turbulence is investigated under the presence of an imposed magnetic field. Such a situation, favorable to the development of anisotropy, is encountered in space plasmas like the solar wind and is well described in physical space by a divergence relation which expresses the statistical conservation of the Elsässer energy flux through the inertial range. The ansatz is made that the development of anisotropy implies a foliation of space correlation. A direct consequence is the possibility to derive a new universal law for the third-order Elsässer moments which is parametrized by the intensity of anisotropy. We use the so-called critical balance condition to fix this parameter and find a unique exact expression. The implication of this new universal vectorial law for solar wind turbulence is eventually discussed.

La turbulence MHD homogène est étudiée en présence d'un champ magnétique extérieur. Une telle situation, favorable au développement d'une dynamique anisotrope, est rencontrée dans les plasmas spatiaux comme le vent solaire et est bien décrit dans l'espace physique par une relation de divergence qui exprime la conservation statistique du flux d'énergie d'Elsässer à travers la zone inertielle. Nous faisons l'ansatz que l'anisotropie implique un feuilletage de l'espace des corrélations. Une conséquence directe est la possibilité d'obtenir une nouvelle loi universelle pour les moments d'Elsässer d'ordre trois qui est paramétrisée par l'intensité de l'anisotropie. Nous utilisons la condition d'équilibre critique pour fixer ce paramètre et trouver une expression unique exacte. L'implication de cette nouvelle loi vectorielle universelle pour le vent solaire est finalement discutée.

1. Introduction

Despite its large number of applications such as climate, atmospherical flows or space plasmas, turbulence is still today one of the least understood phenomena in classical physics; for that reason any exact results appear extremely important [13]. The Kolmogorov's four-fifths (K41) law [29] is often considered as the most important result in three-dimensional (3D) homogeneous isotropic turbulence: it is an exact and nontrivial relation derived from Navier-Stokes equations which implies the third-order longitudinal structure function. When isotropy is *not* assumed the primitive form of the K41 law is the divergence equation [36]

$$-\frac{1}{4}\nabla_{\mathbf{r}} \cdot \mathbf{F^{HD}}(\mathbf{r}) = \varepsilon \,, \tag{1}$$

where ε is the mean energy dissipation rate per unit mass, $\mathbf{r}$ is the separation vector, $\mathbf{F^{HD}}(\mathbf{r}) = \langle \delta\mathbf{v}\delta\mathbf{v}^2 \rangle$ is the vector third-order moment – which is an energy flux [see *e.g.* 12] – and $\delta\mathbf{v} = \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})$. Relation (1) is nothing else than the expression of the energy flux conservation in the inertial range. Then, the K41 law may be seen as a non trivial consequence of equation (1) when isotropy is assumed; it is written as [29]

$$-\frac{4}{5}\varepsilon r = \langle \delta v_L^3 \rangle \,, \tag{2}$$

where L means the longitudinal direction along $\mathbf{r}$. Few extensions of such a result to other fluids have been made; it concerns *e.g.* scalar passively advected such as the temperature or a pollutant in the atmosphere [44] or space magnetized plasmas described in the framework of magnetohydrodynamics (MHD) [42], electron [19] and Hall [20] MHD.

In this paper we investigate 3D homogeneous incompressible MHD turbulence for which the following divergence relation holds [40]

$$-\frac{1}{4}\nabla_{\mathbf{r}} \cdot \mathbf{F}^{\pm}(\mathbf{r}) = \varepsilon^{\pm} \,, \tag{3}$$

where $\mathbf{F}^{\pm}(\mathbf{r}) = \langle \delta\mathbf{z}^{\mp}(\delta\mathbf{z}^{\pm})^2 \rangle$, $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}$ are the Elsässer fields and $\varepsilon^{\pm}$ are the mean Elsässer energy dissipation rates per unit mass. When isotropy is assumed we obtain the exact law for 3D MHD [42]

$$-\frac{4}{3}\varepsilon^{\pm} r = \langle \delta z_L^{\mp} (\delta\mathbf{z}^{\pm})^2 \rangle \,, \tag{4}$$

which may reduce to expression (2) when the magnetic field is taken equal to zero. It is straightforward to demonstrate the compatibility between relations (3) and (4) by performing an integration of the former over a full sphere (ball). The same remark holds for the compatibility between expression (1) and the K41 law.

To date the universal isotropic relations discussed above have never been generalized to 3D homogeneous – non isotropic – turbulence (see however [21–23] for the latest progress). It is basically the goal of this paper to demonstrate rigorously that such a universal relation may be derived in the case of MHD.

2. Impact of a mean magnetic field

The influence of a large-scale magnetic field $\mathbf{B_0}$ on the nonlinear MHD dynamics has been widely discussed during the last fifteen years. The first heuristic picture of MHD turbulence proposed by Iroshnikov-Kraichnan [27, 30] has been criticized and, nowadays, we know that under the presence of $\mathbf{B_0}$ we find turbulent fluctuations with larger fluctuating components in the direction transverse to $\mathbf{B_0}$ than along it, as well as different type of correlations along $\mathbf{B_0}$ and transverse to it [see *e.g.* 1, 2, 6–8, 11, 14, 18, 37, 39]. In other words, the nonlinear transfer occurs differently according to the direction considered with a weaker non linear transfer along $\mathbf{B_0}$ than transverse to it, with possibly different power law energy spectra. One of the most important concept introduced in the last years is the possible existence a critical balance between the nonlinear eddy-turnover time and the Alfvén time [25]. The former time may be associated to the distortion of wave packets whereas the latter may be seen as the duration of interaction between two counter-propagating Alfvén wave packets. A direct consequence of the critical balance is the existence of a relationship (in the inertial range) between length-scales along ($\|$) and transverse ($\perp$) to $\mathbf{B_0}$ [see also 16]. This relation, generally written in Fourier space, is

$$k_\| \sim k_\perp^{2/3} \,. \tag{5}$$

In practice, numerical evidences of relation (5) may be found by looking at the parallel and perpendicular (to $\mathbf{B_0}$) intercepts of the surfaces of constant energy, either in physical space with second-order correlation functions [11, 33] or in Fourier space with spectra [6]. Note that one generally takes a local definition for $k_\|$ by using the local mean magnetic field but it has been shown that a global definition (with the parallel direction along $\mathbf{B_0}$) works quite well if B_0 is strong enough [see *e.g.* 6]. Despite the limitation of direct numerical simulations a scaling relation between parallel and perpendicular length scales seems to emerge whose power law relation is compatible with the critical balance relation (5). Therefore, the idea of a general relationship between length scales during the nonlinear transfer (of energy) from large to small scales may be seen as a natural constrain for theoretical models. We basically use this constrain to solve equation (3) in the case of axisymmetric turbulence.

At this level of discussion, it is important to remark that the assumption of isotropy made to derive the exact law (4) is questionable in the sense that we never observe exactly isotropy. For example in [34, 38] it was shown numerically that despite the absence of a uniform magnetic field ($B_0 = 0$) deviations from isotropy are observed locally with the possibility to get a scaling relation between length-scales along and transverse to the local magnetic field. This local anisotropy is expected to be stronger at larger (magnetic) Reynolds numbers for which the exact law (4) is derived. Therefore, this exact law (4) should be seen as a first order description of MHD turbulence when $B_0 = 0$. More precisely in the derivation of this law one should consider the decomposition

$$\mathbf{F}^{\pm}(\mathbf{r}) = \mathbf{F}^{\pm}_{\mathbf{iso}}(\mathbf{r}) + \delta \mathbf{F}^{\pm}_{\mathbf{ani}}(\mathbf{r}) \,, \tag{6}$$

where the first term in the RHS is the isotropic contribution to the vector third-order moment whereas the second term measures the deviation from isotropy. When the second term is of second order in important then $\delta \mathbf{F}^{\pm}_{\mathbf{ani}} \ll \mathbf{F}^{\pm}_{\mathbf{iso}}$ and the integration of relation (3) over a full sphere – with the application of the divergence theorem – gives the universal law (4).

The derivation of a universal law from equation (3) in the general case of non isotropic turbulence is far from obvious. For example, one needs to find a volume $\mathcal{V}$ such that at its surface $\mathcal{S}$ the normal component F_n of $\mathbf{F}$ is conserved. Then, one can perform an integration of equation (3) over this volume, apply the divergence theorem and obtain a simple expression independent of any parameter (see also Appendix B). In practice, that means one starts with

$$-\frac{1}{4} \int \int \int_{\mathcal{V}} \nabla_{\mathbf{r}} \cdot \mathbf{F}^{\pm}(\mathbf{r}) d\mathcal{V} = \varepsilon^{\pm} \int \int \int_{\mathcal{V}} d\mathcal{V} \,, \tag{7}$$

which gives by the divergence theorem and after integration over the volume

$$-\frac{1}{4} \int \int_{\mathcal{S}} \mathbf{F}^{\pm}(\mathbf{r}) \cdot d\mathcal{S} = \varepsilon^{\pm} \, \mathcal{V} \,, \tag{8}$$

and after projection on the surface vector $d\mathcal{S}$

$$-\frac{1}{4} \int \int_{\mathcal{S}} F^{\pm}_{n}(\mathbf{r}) d\mathcal{S} = \varepsilon^{\pm} \, \mathcal{V} \,. \tag{9}$$

If one assumes that $F_n^{\pm}(\mathbf{r})$ is constant on $\mathcal{S}$ then one obtains

$$-\frac{1}{4}F_n^{\pm}(\mathbf{r}) \int\int_{\mathcal{S}} d\mathcal{S} = -\frac{1}{4}F_n^{\pm}(\mathbf{r})\mathcal{S} = \varepsilon^{\pm}\,\mathcal{V}\,, \tag{10}$$

which leads to the universal law

$$F_n^{\pm}(\mathbf{r}) = -4\varepsilon^{\pm}\,\frac{\mathcal{V}}{\mathcal{S}}\,. \tag{11}$$

The existence of such a volume $\mathcal{V}$ is still an open question and it is likely that it does not exist. However, it is important to note that there exists an infinity of mathematical solutions of equation (3) but they depend on parameters which render the solutions non universal. For example we may have [40]

$$\mathbf{F}^{\pm}(\mathbf{r}) = -4\varepsilon(A^{\pm}\rho\,\mathbf{e}_{\rho} + (1-2A^{\pm})z\,\mathbf{e_z})\,, \tag{12}$$

where ρ and z are the cylindrical coordinates, and $\mathbf{e}_{\rho}$ and $\mathbf{e_z}$ are the corresponding unit vectors (with $\mathbf{e_z} \equiv \mathbf{B_0}/B_0$). Note that the choice $A^{\pm} = 1/2$ gives the universal law for two-dimensional isotropic MHD turbulence, whereas $A^{\pm} = 1/3$ leads to a radial vector and corresponds to the three-dimensional isotropic law [42]. Then, we may expect that relation (12) describes correctly anisotropic MHD turbulence when $A^{\pm} \in [1/3; 1/2]$ with stronger anisotropy when $A^{\pm}$ is closer to 1/2. However, relation (12) does not satisfy the critical balance relation (5) for any values of $A^{\pm}$: indeed, for isotropic turbulence the energy flux vector is radial which expresses the fact that energy cascades radially, whereas when a mean magnetic field is present it is not the case anymore and iso-contours of spectral energy are elongated in the perpendicular direction according to the power law (5) with an elongation more pronounced at small length scales (which means, in the correlation space, an elongation along the mean magnetic field direction). According to relation (12), we see that for a given distance r the energy flux ratio between a point along $\mathbf{e_z}$ and another point along $\mathbf{e}_{\rho}$ is equal to the following constant

$$\frac{F^{\pm}(r\mathbf{e_z})}{F^{\pm}(r\mathbf{e}_{\rho})} = \frac{1-2A^{\pm}}{A^{\pm}}\,. \tag{13}$$

This constant can be very small (when $A^{\pm}$ is close to 1/2) but its precise value does not change the nature of the relation between these two fluxes which is linear. Therefore, it can only lead to a linear law dependence between the parallel and perpendicular intercepts of the surfaces of constant energy (the form of these surfaces being directly related to the intensity and direction of the energy flux). Note that if one considers a slightly different situation with points close to the $\mathbf{e}_\rho$ and $\mathbf{e_z}$ directions with energy fluxes $F^{\pm}(r\mathbf{e}_\rho + \epsilon\mathbf{e_z})$ and $F^{\pm}(\epsilon\mathbf{e}_\rho + r\mathbf{e_z})$ respectively (where ϵ is a small parameter), the conclusion does not change drastically as long as $r \gg \epsilon$; when r becomes of the order of ϵ then both energy flux vectors deviate significantly from the $\mathbf{e}_\rho$ and $\mathbf{e_z}$ directions which does not help for increasing anisotropy at small length scales which needs to have energy flux vectors preferentially along $\mathbf{e}_\rho$.

In order to recover an anisotropic law of the type of (5) – which is a power law – it is necessary to reinforce the energy flux in the $\mathbf{e}_\rho$ direction at small length scales. Then, the following statement is made that the energy flux vector has an orientation closer to the $\mathbf{e}_\rho$ direction when the length scale decreases. This variation must have a power law dependence (with power law index n) in the length scale in order to be compatible with relation (5) which is also a power law. The value of n compatible with the index 2/3 in relation (5) may be determined with critical balance arguments (see Section). We will see that if we incorporate such a requirement in the analysis then we may derive a universal law which, therefore, does not depend on any (non physical) parameter. In practice, the energy flux vectors will belong to an axisymmetric surface $\mathcal{S}_n$ in the three-dimensional space correlation (which means that $\mathbf{F}^{\pm}(\mathbf{r})$ is tangent to $\mathcal{S}_n$ for any points $M' \in \mathcal{S}_n$; see Section and Fig. 1). The manifold $\mathcal{S}_n$ is defined in such a way that the energy flux vectors tend to be perpendicular to $\mathbf{e_z}$ when the distance separation goes to zero which means that turbulence tends to be bi-dimensional at small scales. As we will see in Section , the expected constant -2 for two-dimensional MHD turbulence is indeed recovered from the universal law when the small scale limit is taken.

3. Foliation of space correlation

From several theoretical and numerical analyses we know that MHD turbulence under the influence of $\mathbf{B_0}$ develops anisotropy that increases as the length scale decreases. Additionally, the rms fluctuations at a given separation distance r are more intense when $\mathbf{r}$ is perpendicular to $\mathbf{B_0}$ than when $\mathbf{r}$ is parallel to $\mathbf{B_0}$. This property can be understood as a consequence of the critical balance relation (5) which provides a relationship between the length scales of the fluctuations parallel and perpendicular to the mean magnetic field. Following these considerations and those exposed at the end of Section , we make the ansatz that the energy flux vectors belong to two-dimensional surfaces $\mathcal{S}_n$ in the three-dimensional space correlation (which means that $\mathbf{F}^{\pm}(\mathbf{r})$ is tangent to $\mathcal{S}_n$ for any points $M' \in \mathcal{S}_n$; see Fig. 1). Since the problem is axisymmetric, the manifolds $\mathcal{S}_n$ must be of revolution about the (Mz) axis (with $\mathbf{e_z} \equiv \mathbf{B_0}/B_0$; see Fig. 1). It is defined in such a way that the direction of $\mathbf{F}^{\pm}(\mathbf{r})$ tends to become perpendicular to $\mathbf{e_z}$ when the distance separation r goes to zero. This variation of direction for $\mathbf{F}^{\pm}(\mathbf{r})$ should have a power law dependence in the length scale. Then, the axisymmetric manifold $\mathcal{S}_n$ is defined by the following function

$$z = f_n(\rho) = \rho_0 \left(\frac{\rho}{\rho_0}\right)^n . \tag{14}$$

It is the simplest algebraic function satisfying the conditions $f_n(\rho) \to 0$ when $\rho \to 0$ with a simple power law dependence between ρ and z. Other (exponential or logarithmic) functions may lead to a more complex form with possible trouble to satisfy the previous condition. Without loss of generality we may already note that n must be greater than one to satisfy the anisotropic property (the energy flux vector getting perpendicular to $\mathbf{B_0}$ at small separation distance r). Finally, note that ρ_0 is the value of ρ for which the angle between $\mathbf{r}$ and $\mathbf{e_z}$ is $\pi/4$; therefore ρ/ρ_0 may be seen as a way to delimit the correlation space into two domains where the direction of the separation vector $\mathbf{r}$ is closer to the transverse plane (xMy) or to the parallel direction $\mathbf{e_z}$ (see Fig. 1).

At this point it is important to emphasize that the critical balance measured in MHD turbulence (with $B_0 > 0$) is a situation towards which the nonlinear dynamics converges: it is the main state of the dynamics. In other words, deviations from this state may be found but are of second order in importance. In the same way, the assumption of a foliation of the space correlation (with relation (14)) means that one should write

$$\mathbf{F}^{\pm}(\mathbf{r}) = \mathbf{F}^{\pm}_{\mathbf{fol}}(\mathbf{r}) + \delta\mathbf{F}^{\pm}_{\mathbf{nonfol}}(\mathbf{r}) , \tag{15}$$

where the first term in the RHS is the vector third-order moment which belongs to the foliated space correlation whereas the second term corresponds to other vector contributions which are assumed (ansatz) of second order in importance, namely $\delta\mathbf{F}^{\pm}_{\mathbf{nonfol}} \ll \mathbf{F}^{\pm}_{\mathbf{fol}}$. We will see that the consequence of this ordering is that the universal vectorial law derived below implies correlation between any points in the 3D correlation space (point transverse and parallel to $\mathbf{e_z}$ are reached asymptotically) but not for any directions: for example we cannot have two vectors $\mathbf{F}^{\pm}(M)$ and $\mathbf{F}^{\pm}(M')$ parallel to the axis of symmetry $\mathbf{e_z}$ if M' is close to the (xMy) plane. But it is well-known that the power fluctuations along the z-direction are statistically significantly smaller than those in the transverse direction (see *e.g.* in the solar

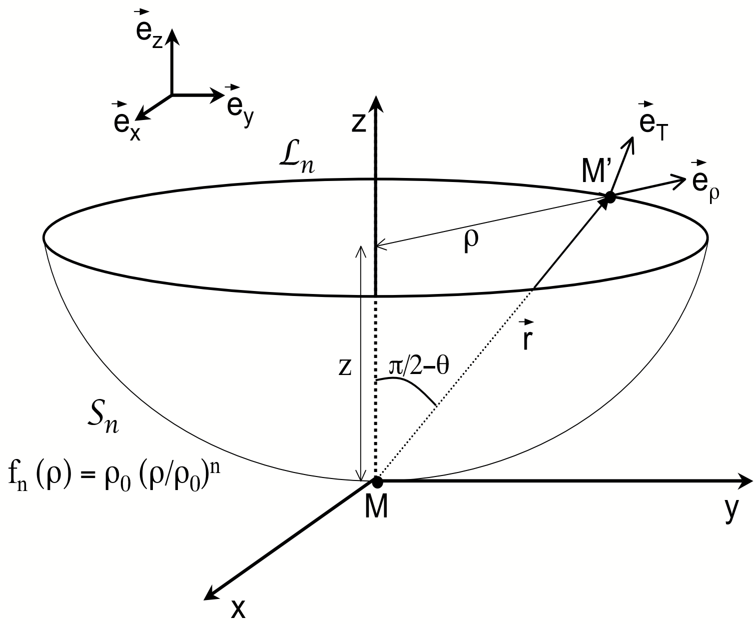


FIG. 1: We perform an integration of relation (3) over the manifold $\mathcal{S}_n$ defined in the half upper space by the function $f_n(\rho) = \rho_0(\rho/\rho_0)^n$ with $n > 1$; note the use of the polar coordinates with $\mathbf{r} = (\rho, z)$. $\mathcal{S}_n$ is a surface of revolution about the (Mz) axis: on this Figure it appears as a "bowl" of axis of symmetry (Mz). The vector $\mathbf{e_T}$ at point M' is tangent to the surface $\mathcal{S}_n$ and perpendicular to the circle $\mathcal{L}_n$ of radius ρ which has also (Mz) for axis of symmetry.

wind where a ratio down to 1/30 may be found for the power magnetic field fluctuations [28]). The non presence of such type of correlations simply means that it is a second order in importance.

Equation (3) is integrated over the manifold $\mathcal{S}_n$ of axis of symmetry (Mz). An illustration is given in Fig. 1 where $\mathcal{S}_n$ appears as a "bowl". It gives

$$-4\varepsilon^{\pm} \int\int_{\mathcal{S}_n} d\mathcal{S}_n \;=\; \int\int_{\mathcal{S}_n} \nabla_{\mathbf{r}} \cdot \mathbf{F}^{\pm}(\mathbf{r})\, d\mathcal{S}_n \,. \tag{16}$$

By the Green's flux theorem (see Appendix A and B) and after integration over the surface, we obtain

$$-4\varepsilon^{\pm}\mathcal{S}_n \;=\; \oint_{circle} \mathbf{F}^{\pm}(\mathbf{r}) \cdot d\mathcal{L}_n \,, \tag{17}$$

where the line integral is performed along a circle $\mathcal{L}_n$ of radius ρ and of axis of symmetry

(Mz). On the example given in Fig. 1, it corresponds to the upper boundary of the "bowl". Note that $d\mathcal{L}_n$ is an elementary vector which is normal to the circle $\mathcal{L}_n$ and tangent to the surface $\mathcal{S}_n$ (see Appendix B). Then, one gets after projection

$$-4\varepsilon^{\pm}\mathcal{S}_n \;=\; \oint_{circle} F_T^{\pm}(\mathbf{r})d\mathcal{L}_n\,, \tag{18}$$

where T means the tangent direction at point M' (see Fig. 1). The problem being axisymmetric, $F_T(r)$ is unchanged along the circle $\mathcal{L}_n$ of axis of symmetry (Mz); then we have

$$-4\varepsilon^{\pm}\mathcal{S}_n \;=\; F_T^{\pm}(\mathbf{r})\oint_{circle} d\mathcal{L}_n = F_T^{\pm}(\mathbf{r})\,2\pi\rho\,, \tag{19}$$

and thus

$$-\frac{4\varepsilon^{\pm}\mathcal{S}_n}{2\pi\rho} = F_T^{\pm}(\mathbf{r})\,. \tag{20}$$

If we introduce the unit vector $\mathbf{e_T}$ along the T–direction we obtain the vectorial relation

$$-\frac{2\varepsilon^{\pm}\mathcal{S}_n}{\pi\rho}\,\mathbf{e_T} = \mathbf{F_T^{\pm}}(\mathbf{r})\,, \tag{21}$$

with

$$\begin{aligned}\mathbf{e_T} \;&=\; \frac{\mathbf{e}_\rho + f_n'(\rho)\mathbf{e_z}}{\sqrt{1+f_n'(\rho)^2}} = \frac{\mathbf{e}_\rho + n(\rho/\rho_0)^{n-1}\mathbf{e_z}}{\sqrt{1+n^2(\rho/\rho_0)^{2(n-1)}}} \\ &=\; \frac{\mathbf{e}_\rho + n\tan\theta\mathbf{e_z}}{\sqrt{1+n^2\tan^2\theta}}\,,\end{aligned} \tag{22}$$

where θ is the angle between $\mathbf{r}$ and the (xMy) plane (see Fig. 1). Note that for the foliated space correlation defined with relation (14) the general form of the divergence operator is

$$\nabla\cdot\mathbf{F} \equiv \frac{1}{\rho}\frac{\partial(\rho F_T)}{\partial T} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}\,, \tag{23}$$

where ϕ is the angle defined in cylindrical coordinates (note that by symmetry $F_\phi = 0$) and dT is the unit length along the tangent direction (see Fig. 1). The surface $\mathcal{S}_n$ for a given ρ is defined as

$$\begin{aligned}\mathcal{S}_n \;&=\; \int 2\pi\rho\,dT = \int_0^\rho 2\pi\rho\sqrt{1+f_n'(\rho)^2}d\rho \\ &=\; \int_0^\rho 2\pi\rho\sqrt{1+n^2\left(\frac{\rho}{\rho_0}\right)^{2(n-1)}}d\rho \\ &=\; \frac{\pi\rho_0^2}{n^{2/(n-1)}}\int_0^X \sqrt{1+X^{n-1}}dX\,,\end{aligned} \tag{24}$$

with

$$X = n^{2/(n-1)} \left(\frac{\rho}{\rho_0}\right)^2 = \left(\frac{nz}{\rho}\right)^{2/(n-1)} = (n \tan\theta)^{2/(n-1)} \,. \tag{25}$$

The combination of the different expressions gives eventually the following exact vectorial law

$$-2\frac{I(X)}{X}\varepsilon^{\pm}\rho\,\mathbf{e_T} = \mathbf{F}^{\pm}_{\mathbf{T}}(\mathbf{r})\,, \tag{26}$$

where

$$I(X) = \int_0^X \sqrt{1+X^{n-1}}dX\,. \tag{27}$$

4. Critical balance condition

The exact vectorial relation (26) implies a parameter n that has to be determined. We shall fix n by a dimensional analysis based on the critical balance condition [25]. To investigate this idea we will restrict our analysis to the inviscid, stationary MHD equations since basically we want an interpretation of the exact relation valid in the inertial range; we thus obtain

$$\mathbf{z}^{\mp}\cdot\nabla\,\mathbf{z}^{\pm} = -\nabla P_* \pm B_0\partial_{\|}\mathbf{z}^{\pm}\,, \tag{28}$$

where P_* is the total pressure. By first noting that the divergence operator applied to (28) allows us to link the total pressure to the left hand side term, and second that $z^+ \sim z^-$ for small cross-correlation; we then arrive to the nontrivial critical balance

$$z_r^{\pm}\nabla_r \sim B_0\partial_{\|}\,, \tag{29}$$

which may also be written as

$$\frac{z_r^{\pm}}{B_0} \sim \frac{\partial_{\|}}{\nabla_r} \sim \frac{k_{\|}}{k_r} = \sin\theta\,, \tag{30}$$

where θ is also the angle between the separation vector $\mathbf{r}$ and the (xMy) plane (see Fig. 1). As we see, relation (30) offers a direct evaluation of the $\mathbf{r}$–direction: therefore, although the external magnetic field does not enter explicitly in the vectorial relation (26), it constrains – as expected – the direction along which the scaling law applies. If we now come back to relation (26), we may write (at first order for small length scales) the dimensional relation which is independent of n

$$z_r^{\pm} \sim (\varepsilon^{\pm}\rho)^{1/3}\,, \tag{31}$$

and obtain

$$\sin\theta \sim \frac{(\varepsilon^{\pm}\rho)^{1/3}}{B_0}\,. \tag{32}$$

In other words, this result means that the scaling relation depends on the strength of the external magnetic field with an orientation close to the (xMy) plane for strong B_0, but also on the scales itself with a direction getting closer to the (xMy) plane at small scales (small r). This dimensional analysis will be used below to derive the unique expression of the vectorial law for anisotropic MHD turbulence since relation (32) gives the following dimensional small-scale constrain

$$\sin\theta \sim \frac{(\varepsilon^{\pm}\rho)^{1/3}}{B_0} \sim \left(\frac{\rho}{\rho_0}\right)^{n-1}, \tag{33}$$

which leads to

$$n = 4/3\,. \tag{34}$$

Note that for other types of fluids the value of n may be different. For example, in electron MHD one finds $n = 5/3$ [23].

5. Universal vectorial law

Following the critical balance idea we shall rewrite expression (26) for $n = 4/3$ which gives

$$-g(\theta)\varepsilon^{\pm} r\,\mathbf{e_T} = \mathbf{F}^{\pm}_{\mathbf{T}}(\mathbf{r})\,, \tag{35}$$

with $g(\theta) \equiv 2\cos\theta I(X)/X$,

$$X = \left(\frac{4}{3}\tan\theta\right)^6, \;\; \mathbf{e_T} = \frac{\mathbf{e}_\rho + (4/3)\tan\theta\mathbf{e_z}}{\sqrt{1+(4/3)^2\tan^2\theta}}\,, \tag{36}$$

and

$$\begin{aligned} I(X) &= \int_0^X \sqrt{1+X^{1/3}}\,dX \\ &= -\frac{16}{35} + \frac{6}{7}\left(1+X^{1/3}\right)^{3/2} X^{2/3} \\ &- \frac{24}{35}\left(1+X^{1/3}\right)^{3/2} X^{1/3} + \frac{16}{35}\left(1+X^{1/3}\right)^{3/2}\,. \end{aligned} \tag{37}$$

It is the main result of the paper. We see that the exact vectorial law has a form close to the isotropic case (4) with a scaling linear in r. However, we observe a θ-angle dependence which reduces the degree of universality of the law. From an observational point of view this prediction turns out to be interesting since in the solar wind the measurements are naturally made at a given angle. Numerical estimate of the function $g(\theta)$ gives a slight variation from 2 to 16/7 for respectively $\theta = 0$ to $\pi/2$. It is important to remark that this law is valid for *any* r and θ which means that we may describe the entire space. Note that the exact law does not imply directional energy dissipation rates per unit mass (like $\varepsilon_{\perp}^{\pm}$ or $\varepsilon_{\parallel}^{\pm}$) which makes a difference with other types of exact results like in wave turbulence where the spectra may be expressed in terms of directional energy transfer fluxes [see *e.g.* 15, 17, 24].

The exact vectorial law is derived by assuming the existence of an external uniform magnetic field (greater than the fluctuations). The extension of this law to a local analysis for which anisotropy is due to a local magnetic field might also be considered but then it is only an approximate law since in our derivation we have considered the entire inertial range.

5.1. Small θ limit

The first interesting limit to analyze is the one for which the energy flux vector is mainly transverse, *i.e.* for small θ. In the limit of small angle, we obtain after a Taylor expansion

$$I(X) \simeq X + \frac{3}{8} X^{4/3} \,, \tag{38}$$

and then after substitution

$$-2\left(1 + \frac{2}{3}\tan^2\theta\right) \varepsilon^{\pm} \rho \, \mathbf{e_T} \simeq \mathbf{F_T^{\pm}}(\mathbf{r}) \,. \tag{39}$$

This relation tends asymptotically to the scaling prediction for 2D MHD turbulence which may be obtained directly after integration (and application of the Green's flux theorem) of expression (3) over a disk with only transverse fluctuations.

5.2. Large θ limit

The second interesting limit for which the exact vectorial law simplifies is the one for which the energy flux vector is mainly parallel to $\mathbf{e_z}$, *i.e.* for large (close to $\pi/2$) θ-angle. In the limit of large angle, we obtain after expansion

$$I(X) \simeq \frac{6}{7} X^{7/6} - \frac{24}{35} X^{5/6} \,, \tag{40}$$

and then after substitution

$$-\frac{16}{7}\left(1-\frac{9}{20}\frac{1}{\tan^2\theta}\right)\varepsilon^{\pm} z\, \mathbf{e_T} \simeq \mathbf{F}^{\pm}_{\mathbf{T}}(\mathbf{r})\,. \tag{41}$$

6. Discussion and conclusion

The interplanetary medium is probably the best example of application of the new universal law. Indeed, it is a medium permeated by the solar wind, a highly turbulent and anisotropic flow which carries the solar magnetic field [see *e.g.* 5, 9, 26, 28]. Several recent works have been devoted to the analysis of low frequencies solar wind turbulence in terms of structure functions by using the exact isotropic law [see *e.g.* 43]. A direct evidence for the presence of an inertial energy cascade in the solar wind is claimed but the comparison between data and theory is moderately convincing because of the narrowness of the inertial range measured. Some recent improvements have been obtained by using a model of the isotropic law where compressible effects are included [10]. Even if the result seems to be better the hypothesis of isotropy is a serious default. Other applications of the MHD laws (exact or modeled) are also found in order for example to evaluate the local solar wind heating [31, 32] along or transverse to the mean magnetic field.

Direct numerical simulations are very important to check for example the applicability of the exact laws discussed in the present paper since there are exact as long as the hypotheses are satisfied. For example, in the isotropic case it is interesting to note that the constant has never been checked – only the power law. Therefore, we are not yet at the same degree of achievement reached for the four-fifth's law for which the constant has been recovered experimentally [3]. Then, for the universal vectorial law derived in this paper it is fundamental to check not only the power law dependence (actually, a first analysis at moderate numerical resolution of 256^3 shows a relatively good agreement with the scaling prediction) but also – and more importantly – the coefficient $g(\theta)$ which is around 2. Only massive numerical simulations like in [35] will allow to take up this challenge.

The interplanetary medium is an excellent laboratory to test new ideas in turbulence. In that respect, it would be interesting to extend the present work to other invariants like the cross-correlation. Recent works have been devoted to this problem where the idea of a dynamic alignment between the velocity and the magnetic field fluctuations has emerged [8] but the confrontation with solar wind data is still not totally convincing [41]. It is the purpose of the author to investigate this problem in a near future in the context of exact laws. Since most of astrophysical space plasmas evolve in a medium where a magnetic field is present on the largest scale of the system the present universal law has potentially a lot of other applications.

Acknowledgments

I acknowledge financial support from Institut universitaire de France.

Appendix A: Green's theorem

Appendix A is devoted to the Green's theorem in two dimensions. Let us consider an oriented plane curve $\mathcal{C}$ and a plane vector field $\mathbf{F}$ defined along $\mathcal{C}$. Then the work of $\mathbf{F}$ along $\mathcal{C}$ is the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{t} \, d\ell \,, \tag{42}$$

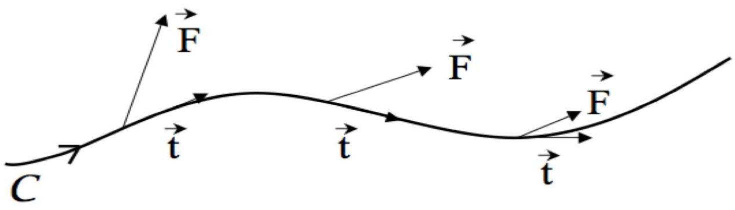
where $\mathbf{t}$ is a unit vector tangent to the curve $\mathcal{C}$ (see Fig. 2; top) and $d\ell$ is an elementary length of curve $\mathcal{C}$.

If now $\mathcal{C}$ is a closed curve enclosing a region S in the plane, counterclockwise (see Fig. 2; bottom) and if $\mathbf{F}$ is defined in the plane (on $\mathcal{C}$ and also in S), then we have the relation

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{t} \, d\ell = \int \int_S \nabla \times \mathbf{F} \, dS \,. \tag{43}$$

which means that the work of $\mathbf{F}$ along a closed integral line is equal to the sum of the curl of $\mathbf{F}$ on the surface S. It is the Green's theorem.

A short proof of the Green's theorem comes as follows. Let us consider the particular case of a rectangular closed curve ABCDA whose orientation defines the x and y directions.



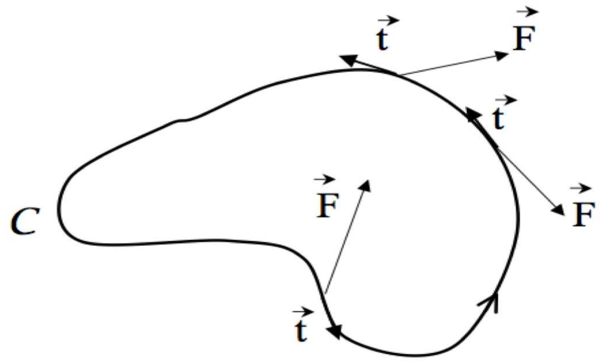


FIG. 2: Top: Oriented plane curve $\mathcal{C}$ along which the work of $\mathbf{F}$ is computed. The tangent vector $\mathbf{t}$ is oriented according to the path of integration namely, here, from left to right. Bottom: Oriented plane curve $\mathcal{C}$ that encloses a region S.

On the one hand, one has

$$\begin{aligned}
\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{t}\,d\ell &= \oint_{\mathcal{C}} \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \cdot \begin{Bmatrix} dx \\ dy \end{Bmatrix} && (44)\\
&= \int_A^B F_x(x,y_1)dx + \int_B^C F_y(x_2,y)dy && (45)\\
&+ \int_C^D F_x(x,y_2)dx + \int_D^A F_y(x_1,y)dy && (46)\\
&= \int_{x_1}^{x_2} (F_x(x,y_1) - F_x(x,y_2))dx && (47)\\
&+ \int_{y_1}^{y_2} (F_y(x_2,y) - F_y(x_1,y))dy\,. && (48)
\end{aligned}$$

On the other hand, one has

$$\int\int_S \nabla \times \mathbf{F}\, dS = \int_{x_1}^{x_2}\int_{y_1}^{y_2}(\partial_x F_y - \partial_y F_x)dxdy \tag{49}$$

$$= \int_{y_1}^{y_2}(F_y(x_2, y) - F_y(x_1, y))dy \tag{50}$$

$$- \int_{x_1}^{x_2}(F_x(x, y_2) - F_x(x, y_1))dx\,, \tag{51}$$

which is equal to the work.

Appendix B: Green's flux theorem

This second appendix is devoted to the Green's flux theorem which may be seen as the two-dimensional version of the well-known divergence theorem. It is also called the Normal form of Green's theorem. Let us consider an oriented plane curve $\mathcal{C}$ and a plane vector field $\mathbf{F}$ defined along $\mathcal{C}$. Then the flux of $\mathbf{F}$ across $\mathcal{C}$ is the line integral

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\, d\ell\,, \tag{52}$$

where $\mathbf{n}$ is the unit vector normal to the curve $\mathcal{C}$ pointing 90 degrees clockwise from the tangent direction of $\mathcal{C}$ (see Fig. 3) and $d\ell$ is an elementary length of curve $\mathcal{C}$.

If now $\mathcal{C}$ is a curve that encloses a region S counterclockwise (see Fig. 3; bottom) and if $\mathbf{F}$ is defined in the plane (on $\mathcal{C}$ and also in S), then we have the relation

$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\, d\ell = \int\int_S \nabla\cdot\mathbf{F}\, dS\,, \tag{53}$$

which means that the flux of $\mathbf{F}$ across a closed integral line is equal to the sum of the divergence of $\mathbf{F}$ on the surface S. It is the Green's flux theorem.

A short proof of the Green's flux theorem comes as follows. Let us consider the same particular case as in the first appendix of a rectangular closed curve ABCDA whose orientation

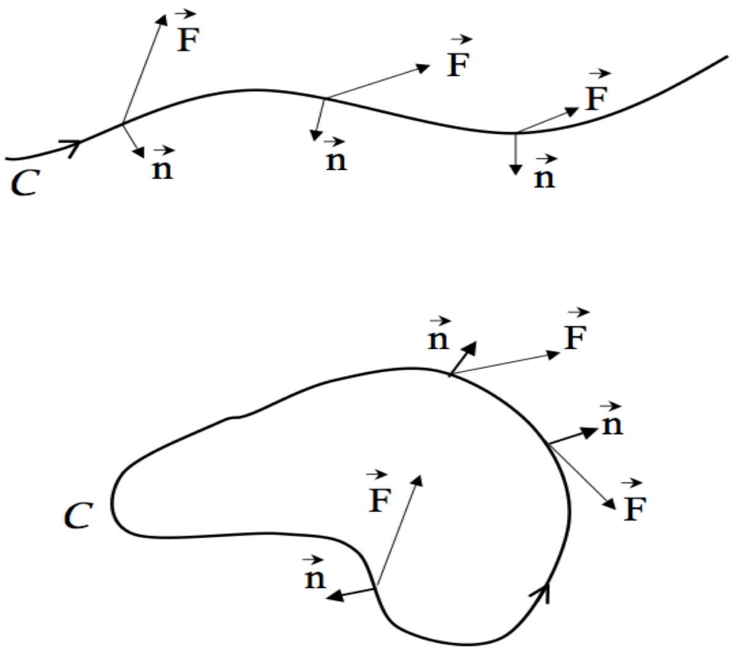


FIG. 3: Top: Oriented plane curve $\mathcal{C}$ across which the flux of $\mathbf{F}$ is computed. The normal direction is oriented 90 degrees clockwise from the tangent direction. Bottom: Oriented plane curve $\mathcal{C}$ that encloses a region S.

defines the x and y directions. On the one hand, one has

$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,d\ell = \oint_{\mathcal{C}} \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \cdot \begin{Bmatrix} dy \\ -dx \end{Bmatrix} \tag{54}$$

$$= -\int_A^B F_y(x,y_1)dx + \int_B^C F_x(x_2,y)dy \tag{55}$$

$$- \int_C^D F_y(x,y_2)dx + \int_D^A F_x(x_1,y)dy \tag{56}$$

$$= -\int_{x_1}^{x_2} (F_y(x,y_1) - F_y(x,y_2))dx \tag{57}$$

$$+ \int_{y_1}^{y_2} (F_x(x_2,y) - F_x(x_1,y))dy\,. \tag{58}$$

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On the other hand, one has

$$\int\int_S \nabla \cdot \mathbf{F}\, dS = \int_{x_1}^{x_2}\int_{y_1}^{y_2} (\partial_x F_x + \partial_y F_y) dx dy \tag{59}$$

$$= \int_{y_1}^{y_2} (F_x(x_2, y) - F_x(x_1, y)) dy \tag{60}$$

$$+ \int_{x_1}^{x_2} (F_y(x, y_2) - F_y(x, y_1)) dx\,, \tag{61}$$

which is equal to the flux. The proofs given in these two appendices may be found in detail in the video lectures given by [4].

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